Multi-Variable Calculus Notes

1. Partial Derivatives:
- Let $F(x,y)$.
  I.e. F is a function of x and y.

  a) When we find the derivative of F with respect to (w.r.t) x, we treat y as a constant.

  b) Likewise, when we find the derivative of F w.r.t y, we treat x as a constant.

- E.g. Let $F(x,y) = x^2 + xy + \frac{y^2}{2}$

  a) Solve $\frac{\partial F}{\partial x}$

  Soln:

  $\frac{\partial F}{\partial x}$ or $\partial_x F$ or $F_x = 2x + y$

  Here, since we're differentiating w.r.t to x, y is treated as a constant.

b) Solve $\frac{\partial F}{\partial y}$

Soln:

$\frac{\partial F}{\partial y}$ or $\partial_y F$ or $F_y = x + y$

Here, since we're differentiating w.r.t. to y, x is treated as a constant.

c) Solve $\frac{\partial^2 F}{\partial x \partial y}$

Soln:

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}(x+y)$$
$$= 1$$

$$\frac{\partial^2 F}{\partial x \partial y} \equiv \partial_{xy} F$$

d) Solve $\frac{\partial^2 F}{\partial y \partial x}$

Soln:

$$\frac{\partial^2 F}{\partial y \partial x} \text{ or } \partial_{yx} F = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$
$$= \frac{\partial}{\partial y}(2x+y)$$
$$= 1$$

Note: $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y \partial x}$ are called mixed partial derivatives and the Mixed Derivative Rule states that $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$.

2. Tangent Lines:

- The equation of a tangent line on a 2-D graph is: $y - y_0 = f'(x_0)(x - x_0)$.

However, we can rewrite the above eqn as: $f(x) = f(x_0) + f'(x_0)\,dx$ where $f(x) = y$, $f(x_0) = y_0$ and $dx = (x - x_0)$.

In calculus, dx means a little bit of x and dy means a little bit of y.

- The eqn of a tangent plane on a 3-D graph is: $F(x,y) = F(x_0, y_0) + F_x(x - x_0) + F_y(y - y_0)$

However, we can rewrite the above eqn as: $F(x,y) = F(x_0, y_0) + F_x\,dx + F_y\,dy$

3. Finding when df and $dF = 0$.

- Consider the eqn

$f(x) = f(x_0) + f'(x)\,dx$, and recall that $dx = x - x_0$.

$$\underbrace{f(x) - f(x_0)}_{df} = f'(x)\,dx$$

Hence, $df = f'(x)\,dx$.

$df = 0$ iff $f'(x)\,dx = 0$.

This means that $df = 0$ iff $f(x)$ is a constant.

- Consider the eqn

$F(x, y) = F(x_0, y_0) + F_x\,dx + F_y\,dy$

Rewriting this eqn, we get

$$\underbrace{F(x, y) - F(x_0, y_0)}_{dF} = F_x\,dx + F_y\,dy$$

Hence, $dF = F_x\,dx + F_y\,dy$

$dF = 0$ iff $F_x\,dx = -F_y\,dy$.

This means that $dF = 0$ iff F is a constant.